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ON STABILIZATION OF THE STEADY STATE ROTATIONS OF A RIGID BODY*

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Theorems on stabilization of nonperturbed motion /1/ are used to construct the control moments ensuring the stability of optimal stabilization of the steady state rotations of a rigid body.

The stability of permanent axes of a rigid body in a central Newtonian force field was investigated in /2,3/. In the case of motion of a body in force fields more general than the central Newtonian field, the set of permanent axes was determined in /4/ and its stability studied in /5/.

Since the system in question is conservative, the steady state rotations are non-asymptotically stable. Their asymptotic stability can be ensured by means of control moments applied to the principal axes of inertia of the rigid body during its motion. In particular, a possibility of stabilizing the motion of a rigid body by means of pendulums was shown in /6/.

Let a rigid body with principal moments of inertia denoted by A, B and C and the center of mass at the point (x_0, y_0, z_0) of a coordinate system the axes of which are directed along the principal axes of inertia of the body, move in a force field admitting a force function $U(\gamma_1, \gamma_2, \gamma_3)$ where γ_i are the direction cosines of the permanent axis in the coordinate system attached to the body. Then the Euler-Poisson equations have a particular solution $p = \omega l_1$, $q = \omega l_2$, $r = \omega l_3$, $\gamma_i = l_i$, where p, q and r are the projections of the vector of instantaneous angular velocity of rotation of the body on the coordinate axes shown above and ω, l_i, are constants.

The equations of perturbed motion in the presence of control moments u_1, u_2 and u_3 about the principal axes of inertia, with $d^2U/d\gamma_i \cdot \partial\gamma_j\ (i \neq j)$, have the form /4/

$$\begin{aligned}
&Ax_1^{\cdot} = (B-C)\,\omega\,(l_3x_2 + l_2x_3) + (l_3u_{22} - u_{30})\,y_2 - (l_2u_{33} - \\
&\quad u_{20})\,y_3 + (B-C)\,x_2x_3 + (u_{22} - u_{33})y_2y_3 + Au_1 + \varepsilon\,(A \to B \to C,\ \ 1 \to 2 \to 3) \\
&y_1^{\cdot} = -l_3x_2 + l_2x_3 + \omega l_3y_2 - \omega l_2y_3 + x_3y_2 - x_2y_3 \\
&u_{i0} = \frac{\partial U}{\partial\gamma_i},\quad u_{ii} = \frac{\partial^2 U}{\partial\gamma_i^2},\quad \gamma_i = l_i
\end{aligned} \tag{1}$$

Here x_i and y_i denote the values of the perturbations in the quantities p, q, r and γ_i respectively, and ε denote the terms of the third and higher orders of smallness relative to the perturbations $y_i\ (i = 1, 2, 3)$.

The equations which have not been written out are obtained by circular transposition of the letters and indices shown in brackets. In the case when the control moments are absent, i.e. $u_i \equiv 0$, equations (1) have the following function as their first integral /5/

$$\begin{aligned}
&V = Ax_1^2 + Bx_2^2 + Cx_3^2 - 2\omega\,(Ay_1x_1 + Bx_2y_2 + Cx_3y_3) + \\
&(\lambda - u_{11})\,y_1^2 + (\lambda - u_{22})y_2^2 + (\lambda - u_{33})y_3^2 + f\,(y_1,\ y_2,\ y_3) \\
&\lambda = A\,\omega^2 - u_{10}l_1^{-1} = B\omega^2 - u_{20}l_2^{-1} = C\omega^2 - u_{30}l_3^{-1}
\end{aligned} \tag{2}$$

where $f\,(y_1, y_2, y_3)$ are forms of the third and higher orders with respect to the perturbations y_i.

In the absence of the control moments $u_i\ (i = 1, 2, 3)$ the form (2) for the stable steady state rotations is a positive definite function of the variables x_i and y_i, and its derivative calculated by virtue of the equations of perturbed motion, is equal to zero /5/.

If, on the other hand, the moments u_i act along the principal axes of inertia of the rigid body, then the form (2) is no longer the integral of the equations of perturbed motion (1) and its derivative computed by virtue of these equations is equal to

$$V^{\cdot} = \sum u_i \frac{\partial V}{\partial x_i} = 2A\,(x_1 - \omega y_1)_{u_1} + 2B\,(x_2 - \omega y_2)\,u_2 + 2C\,(x_3 - \omega y_3)\,u_3$$

We note that in the perturbed motion $(x_i \neq 0,\ y_i \neq 0)$ the relations $x_i \neq \omega y_i\ (i = 1, 2, 3)$ hold for the permanent axis under investigation. Indeed, since the quantities x_i represent the

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perturbed values of the projections of the vector of instantaneous angular velocity on the principal axes of inertia of the rigid body and the quantities y_i are the perturbed values of the direction cosines of the permanent axis relative to the principal axes of inertia of the body, the latter rotating about the permanent axis with a constant angular velocity ω, it follows that the relations $x_i = \omega y_i$ hold only in the case of an unperturbed motion, i.e. $x_i = \omega y_i$ when $x_i = y_i = 0$. In other words, when $x_i = \omega y_i$, then the equations of perturbed motion contain no complete trajectories /1/. This enables us to arrive at the following conclusion. If the control moments are chosen in the form

$$2u_i = -\alpha_i (x_i - \omega y_i) \qquad (i = 1,\ 2,\ 3)$$

where α_i are positive constants, then the derivative of the function (2)

$$V^{\cdot} = -\alpha_1 A\ (x_1 - \omega y_1)^2 - \alpha_2\ B\ (x_2\ - \omega y_2)^2 - \alpha_3 C\ (x_3 -\ \omega y_3)^2$$

computed by virtue of the equations of perturbed motion (1), will be a negative definite function of the variables x_i and y_i, and the steady state rotations of the rigid body will be asymptotically stable in accordance with the Liapunov theorem of asymptotic stability.

Let us now pose the problem of optimal stabilization of the steady state rotations of a rigid body. We shall find such control moments u_i, which will ensure the asymptotic stability of the unperturbed motion $x_i = y_i = 0$ and a minimum of the functional

$$J = \sum \int_0^\infty (\alpha_{ii} x_i^2 + 2\beta_{ii} x_i y_i + \gamma_{ii} y_i^2 + \delta_i u_i^2)\, dt \tag{3}$$

where the integrand expression is a positive definite function of the variables x_i, y_i and u_i.

Let us consider, in accordance with the theorem of optimal stabilization /1/, the positive definite function

$$V^{\cdot} = \Sigma\ (c_{ij} x_i x_j + d_{ij} y_i y_j\ +\ e_{ij} x_i y_j) \tag{4}$$

where c_{ij}, d_{ij} and e_{ij} are constants, and demand that its total derivative computed by virtue of the equations of perturbed motion (1) be equal to the integrand expression of the functional (3), i.e. that the relation

$$\sum [(c_{ij} x_j + e_{ij} y_j)\, x_i^{\cdot} + (d_{ij} y_j + e_{ij} x_i)\, y_i^{\cdot}] = - \sum (\alpha_{ii} x_i^2 + 2\beta_{ii} x_i y_j + \gamma_{ii} y_i^2 + \delta_i u_i^2) \tag{5}$$

where x_i, y_i $(i = 1,\ 2,\ 3)$ are found from the equations (1), holds.

According to the theorem of optimal stabilization, the sum of the left- and right-hand parts of (5) is smallest when the controls are optimal, and this leads to the following equations for determining the control moments:

$$\sum (c_{ij} x_j + e_{ij} y_j) \frac{\partial x_i^{\cdot}}{\partial u_i} + 2\delta_i u_i = \sum (c_{ij} x_j + e_{ij} y_j) + 2\delta_i u_i = 0$$

The above equations yield the control moments acting along the principal axes of inertia of the rigid body in perturbed state, as linear functions of the perturbations

$$2\delta_i u_i = \Sigma\ (c_{ij} x_j + e_{ij} y_j)\ (i = 1,\ 2,\ 3) \tag{6}$$

Substituting the values of the control moments (6) into (5) and equating the coefficients accompanying the same products of the perturbations x_i and y_i in both sides of the equations (5), we obtain a linear system of equations for determining the coefficients of the quadratic form (4). The problem of solvability of the system remains however open. Moreover, after solving this problem in a positive manner, we must check the conditions under which the function (4) will be positive definite. We can however establish the existence of a positive definite function which solves the problem of optimal stabilization of the steady state rotations of a rigid body, by considering the form (2) and the functional

$$J = \int_0^\infty [A^2 \delta_1^{-1} (x_1 - \omega y_1)^2 + B^2 \delta_2^{-1} (x_2 - \omega y_2)^2 +\ C^2 \delta_3^{-1} (x_3 - \omega y_3)^2 + \delta_1 u_1^2 + \delta_2 u_2^2 + \delta_3 u_3^2]\, dt \tag{7}$$

The optimal moments along the principal axes of inertia of the rigid body stabilizing the steady state rotations of this body and minimizing at the same time the functional (7), should be chosen in the form

$$\delta_1 u_1\ =\ -A\ (x_1 - \omega y_1),\ \ \delta_2 u_2\ =\ -B\ (\omega_2 - \omega y_2),\ \ \delta_3 u_3\ =\ -C\ (x_3 - \omega y_3)$$

We note that the optimal stabilization of the steady state rotations of a rigid body can be carried out using the first order approximation /1/.

REFERENCES

1. MALKIN I.G., Theory of Stability of Motion. Moscow, NAUKA, 1966.
2. BELETSKII V.V., Some problems in the motion of a rigid body in a Newtonian force field. PMM Vol.21, No.6, 1957.
3. POZHARITSKII G.K., On the stability of permanent rotations of a rigid body with a fixed point, located in a Newtonian central force field. PMM Vol.23, No.4, 1959.
4. APYKHTIN N.G., Permanent axes of rotation of a rigid body with a fixed point when the integrals of D.N. Goriachev exist. PMM Vol.27, No.5, 1963.
5. APYKHTIN N.G., On stability of certain permanent rotations of a rigid body. PMM Vol.29, No.2, 1965.
6. KREMENTULO V.V., On the optimal stabilization of a rigid body with a fixed point by means of pendulums. PMM Vol.30, No.1, 1966.

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